Quasi-Isotropization of the Inhomogeneous Mixmaster Universe Induced by an Inflationary Process

A. A. Kirillov* and G. Montani**

*)Institute for Applied Mathematics and Cybernetics
10 Ulyanova str., Nizhny Novgorod, 603005, Russia

**)ICRA—International Center for Relativistic Astrophysics
Dipartimento di Fisica (G9), Università di Roma, "La Sapienza", Piazzale Aldo Moro 5, 00185 Rome, Italy

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Abstract

We derive a "generic" inhomogeneous "bridge" solution for a cosmological model in the presence of a real self-interacting scalar field. This solution connects a Kasner-like regime to an inflationary stage of evolution and therefore provides a dynamical mechanism for the quasi-isotropization of the universe. In the framework of a standard Arnowitt-Deser-Misner Hamiltonian formulation of the dynamics and by adopting Misner-Chitrè-like variables, we integrate the Einstein-Hamilton-Jacobi equation corresponding to a "generic" inhomogeneous cosmological model whose evolution is influenced by the coupling with a bosonic field, expected to be responsible for a spontaneous symmetry breaking configuration. The dependence of the detailed evolution of the universe on the initial conditions is then appropriately characterized.

1 Introduction

As is well known [1, 2] (see also [3]-[6]) the general solution of the Einstein equations near a cosmological singularity exhibits an oscillatory stochastic behavior. This feature of the very early universe is in striking contrast with the universe as described by the well-tested theory of the standard cosmological model [7], which is based on the highly symmetric Friedmann-Robertson-Walker geometry. However, the experimental evidence for the homogeneous and isotropic character of our actual universe concerns relatively late stages of evolution. Indeed the good agreement of the light element nucleosynthesis prediction with the observed abundances implies that the standard cosmological model is surely valid after 10^{-3}-10^{-2} sec from the big bang, but says nothing about the very early dynamics before this time.

In this respect, by observing that the Friedmann-Robertson-Walker metric is an unstable solution of the Einstein equations when regarded as running backwards in time [8], then from the existence of structures in the universe [9] like galaxies and clusters of galaxies, we may infer (even in the presence of an inflationary scenario) that such symmetric geometry cannot continue all the way up to the initial singularity. In fact, the clumpiness of the universe indicates the necessity for very early perturbations of its homogeneity and isotropy, which unavoidably "explode" when approaching the big bang[1].

The instability of the FRW metric, when regarded backward in time, means that there exists some moment t_* before which the evolution of the Universe was to be described by a "generic" inhomogeneous model, or , by other words, when the BKL (Belinskii - Khalatnikov - Lifshitz) picture [1, 2] holds. In general this moment represents a free parameter which depends on initial conditions and, in particular, on specific properties of matter. In vacuum case the applicability of the BKL picture was shown [3] to be described by the inequality $L_h \ll L_{in}$ where $L_h \sim t$ is the horizon size and L_{in} is the characteristic scale of inhomogeneity (or, to be more precise, the mean geometrical value of all leading inhomogeneity scales [2])

Thus, in the vacuum case the moment $t_* = t_{in}$ when the Mixmaster phase (i.e. the oscillatory regime) ends corresponds to the situation $L_h \sim L_{in}$, which can be roughly considered as a boundary of the BKL approximation (in the early stages only the stiff matter influences the evolution of the metric). The reversibility of the Einstein equations

[1]Actually, this represents a rather weak point since the universe could start out in a completely homogeneous and isotropic state, while primeval perturbations which necessary for the origin of structures might appear later during the inflationary era.

[2]We recall that inhomogeneity of the metric is described by a set of distinct length scales L^A_{in} which correspond to different terms in the spatial scalar curvature. The anisotropic evolution leads to the increase of some curvature terms (i.e., the corresponding scales decrease with respect to the horizon size) which causes the transition between Kasner regimes as described in Ref. [2]. From the qualitative standpoint, the transition of a Kasner epoch occurs when the horizon size matches the smallest of the scales $L_h \sim L^A_{in}$ and therefore the above inequality is violated at least for one of the scales. However, the duration of such violations are small compared with the duration of Kasner epochs and for the average geometrical value of the scales, this inequality still holds. Anyway the greater the inequality, the better the BKL picture works.

means that the BKL picture works in both directions of time (which is explicitly realized in the so-called billiard representation [3, 4]). The fact that t_{in} is a free parameter means that near the singularity at a moment t_0, the initial conditions can always be chosen in such a way that $t_0 \ll t_{in}$ and there is a defined period (which depends on the degree of inhomogeneity of the metric) when the BKL picture still works even with the increase of time.

We note that there exists another important moment t_m when the matter "switches on" (i.e., starts to influence the evolution of metric). This moment t_m represents another free parameter specified by initial conditions for matter. Therefore, the actual moment t_* when the Mixmaster evolution (BKL picture) breaks down depends on the relation between this two parameters t_m and t_{in}.

In the present paper we consider a restricted region of initial conditions in which the inequality $t_m \ll t_{in}$ holds, which means that the matter starts to dominate deep inside the BKL regime. From the physical standpoint this restriction means that the transition from the one BKL regime to another takes place when locally, i.e., on the scale of causal connection, the universe still looks like a homogeneous model, for the inequality $L_h \sim t \ll t_{in}$ is fulfilled, while global properties are described by a general inhomogeneous metric. Therefore, it is natural to expect that some results obtained for homogeneous models can also be applied there. In particular, if t_m corresponds to the beginning of an inflationary period, then the inequality $L_h \ll t_{in}$ remains valid from the BKL era through the inflationary period [3] and therefore we can use the results of Ref. [10].

The chaotic nature of the evolution (both temporally and spatially) implies that the geometry of the very early universe should be described by a stationary statistical distribution [3] (see also [11] -[14]). Indeed in this context we may speak about geometry only in an average sense; it turns out that mean values of all geometrical quantities (lengths, scalar products, etc.) during the oscillatory regime are unstable (higher moments have the same order of magnitude as the average values) and therefore near the singularity the universe does not possess a stable background. We remark that the same situation holds in the quantum evolution of the inhomogeneous Mixmaster [15, 16] universe, although in the quantum case the statistical distribution has a different (but somehow related) nature.

These considerations first pose the problem of the origin of a stable background and second how this background could arise out of this chaos compatible with the notion of isotropy (on the basis of any acceptable early history of our actual universe). However, the strong correlation between the appearance of a stable background and its isotropic character is a key feature of the very early cosmology. Either on a quantum level or on a classical one the isotropic component of the metric tensor (i.e. the volume of the universe) is a monotonic function of the time variable (which may actually be taken as the time coordinate itself) and therefore does not contain any physical degrees of freedom, which are instead entirely contained in the anisotropic components. In other

[3] We recall that during the inflation the inhomogeneity scales increase more rapidly than does the horizon size.

words a stable background metric can only appear when the anisotropy of the universe is sufficiently suppressed [3, 16].

In vacuum inhomogeneous models this problem was considered first in [17], which outlines how a classical background can arise from the Planckian epoch of the universe essentially when the oscillatory regime is over, i.e. at the moment when the characteristic scale of inhomogeneity L_{in} matches the horizon size L_h. Analogously, in the presence of matter described by vector fields, the background appears when the horizon size reaches the minimal scale between the one related to the vector field and the characteristic scale of inhomogeneity [18] (for a discussion of chaos in superstring cosmology relative to Einstein-dilaton-p-form fields see [19] and [20]). It is important that in both cases the anisotropy of the universe decays (i.e. it becomes smaller and smaller) during the natural Mixmaster-like evolution from the initial singularity (it may be worth noting the analysis presented in [21] on quiescent cosmological singularities).

In this paper we consider the origin of a background space when a real self-interacting scalar field is present in the universe. As we shall see in this case the appearance of a background depends on the initial conditions (to be assigned on a nonsingular space-like hypersurface); in the configuration (phase) space there are two regions of initial conditions for which the evolution behaves in qualitatively different ways.

The first region corresponds to the case when the potential term of the scalar field becomes a dominating term before the end of the Mixmaster evolution ($L_h \sim L_{in}$) (e.g., such a region can be characterized by the inequality $L_c \ll L_{in}$, L_c denoting the Compton length associated with the scalar field, so that the horizon size matches first the Compton scale). In this case the scalar field through its energy completely governs the quasi-isotropization process (i.e. the process which gives origin to a stable background). The appropriate region of initial conditions contains a subregion which corresponds to an inflationary-like evolution of the universe. The second region corresponds to the case when the scalar field potential remains small and, from a qualitative point of view, the origin of a stable background occurs in the same way as in vacuum models.

Below we will consider the first region only, having in mind the idea that a classical quasi-isotropic universe may emerge, up to suitable initial conditions, from general cosmological dynamics, essentially by virtue of an inflationary expansion due to the potential term of the real scalar field. Indeed the main result of this paper is to show the existence of a set of initial conditions of a nonzero measure, corresponding to which the anisotropy of the universe decays exponentially during an inflationary phase (in homogeneous models the inflationary phase and the isotropization of the Universe has been considered in Ref. [10]). Thus the analysis of the "generic" cosmological solution shows how the inflation phenomenon is the "bridge" between the chaoticity near the big bang (indeed in the presence of a real scalar field the Mixmaster contains only a finite number of oscillations [22, 23]) and the phenomenology of the standard cosmological model.

In section 2 we develop the standard Arnowitt-Deser-Misner (ADM) Hamiltonian formulation [24, 25] which is at the foundation of our derivation in section 3 of a "generic" solution of the Einstein-Hamilton-Jacobi equation in presence of a real self-interacting scalar field. Such a solution interpolates between a Kasner-like regime and an inflationary scenario and is to be regarded as the main result of this paper (for a related discussion of the Bianchi I model in the path integral formalism see [26]). In Section 4 we provide a reformulation of the system dynamics in terms of Misner-Chitrè-like variables, in order to give the most appropriate framework for the analysis presented in Section 5 and devoted to emphasize the modification induced in the details of the universe evolution by assigning different initial conditions to the dynamical quantities involved in the problem. Thus this analysis defines the range of existence for the solution we have obtained.

2 Hamiltonian Formulation of the Dynamics

Let us start by fixing the dynamical framework for our investigation of the "generic" inhomogeneous dynamics. We first observe that the dynamical regime we find will be regarded as "generic" in the sense that it possesses the number of physically arbitrary functions of the spatial coordinates (i.e. real degrees of freedom of the physical system) required to specify a generic Cauchy problem on a nonsingular spatial hypersurface (having in mind one which is arbitrarily close to the big bang) for a generic $(n+1)$-dimensional space-time, containing a real self-interacting scalar field, the number of physically independent degrees of freedom is $n(n-1)$. Indeed this number is $(n+1)(n-2)/2$ for the gravitational field, i.e. the number of independent polarizations of a gravitational wave, plus 1 for the real scalar field, but both these fields satisfy second order equations.

The line element of a generic $(n+1)$-dimensional space-time (for the sake of generality we will consider the most general case and the results for our actual universe will follow immediately by setting $n=3$) admits the following standard (ADM) representation:

$$ds^2 = N^2dt^2 - g_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt). \tag{1}$$

Then the Einstein-Hilbert action takes the form (in what follows we use units of the Planck length)

$$I = \int d^nx dt \left\{ \pi^{\alpha\beta}\frac{\partial}{\partial t}g_{\alpha\beta} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N^\alpha H_\alpha \right\}, \tag{2}$$

where the super-Hamiltonian H^0 and the super-momentum H_α are respectively

$$H^0 = \frac{1}{\sqrt{g}}\left\{ \pi^\alpha_\beta \pi^\beta_\alpha - \frac{1}{n-1}\left(\pi^\alpha_\alpha\right)^2 + g\left(W - R\right)\right\}, \tag{3}$$

$$H_\alpha = -2\nabla_\beta \pi^\beta_\alpha + \Pi_\phi \partial_\alpha \phi, \tag{4}$$

where $W(\phi) = \frac{1}{2}\left\{g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right\}$ and R is the spatial scalar curvature constructed from the spatial metric $g_{\alpha\beta}$.

A fundamental step in our investigation consists of rewriting the above (ADM) formulation in a form which is useful for our purposes, but which retains its degree of generality. It turns out to be convenient to use the so-called generalized Kasner-like parameterization of the dynamical variables in terms of n logarithmic scale variables q_a

and n spatial frame covector fields ℓ^a_α dual to a spatial frame consisting of n vectors L^α_a (having inverse component matrices). The n-dimensional metric components and their conjugate momenta are represented in terms of these variables as follows

$$g_{\alpha\beta} = \sum_a \exp\{q^a\}\,\ell^a_\alpha \ell^a_\beta\,, \qquad \pi^\alpha_\beta = \sum_a p_a L^\alpha_a \ell^a_\beta\,, \tag{5}$$

where the covectors ℓ^a_α can contain only $n(n-1)$ arbitrary functions of the spatial coordinates. By definition the Kasner vectors are eigenvectors for both the momentum matrix π^α_β and the metric $g_{\alpha\beta}$ and therefore in the case of a generic point in spacetime, such a decomposition is unique ($n\,(n-1)$ arbitrary functions contained in Kasner vectors ℓ^b_α and $2n$ functions q^a and p_a replace the $n\,(n+1)$ functions contained in $g_{\alpha\beta}$ and π^α_β). This general form of the metric is the most suitable for treating the Kasner-like regime.

A further refinement of the parameterization can be made by separating the different types of contributions to the matrix ℓ^a_α as follows (e.g., see Ref. [2]):

$$\ell^a_\alpha = U^a{}_b S^b_\alpha, \qquad U^a{}_b \in SO(n), \qquad S^a{}_\alpha = \delta^a{}_\alpha + R^a{}_\alpha \tag{6}$$

with $R^a{}_\alpha$ denoting a triangle matrix ($R^a{}_\alpha = 0$ if $a < \alpha$) and, therefore, it contains only $n(n-1)/2$ arbitrary functions of coordinates, while the rest $n(n-1)/2$ arbitrary functions are included in the rotation matrix U^a_b. By substituting (5) and (6) into (2), we rewrite the action functional in the form (any repeated index is to be regarded as summed)

$$I = \int (p_a \frac{\partial q^a}{\partial t} + T^\alpha_a \frac{\partial R^a_\alpha}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \tag{7}$$

where $T^\alpha_a = 2\sum_b p_b L^\alpha_b U^b_a$ and the Hamiltonian and the momentum constraints take the form

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \sum_a p_a^2 - \frac{1}{n-1} (\sum_a p_a)^2 + \frac{1}{2}\Pi^2_\phi + U \right\}, \tag{8}$$

$$H_\alpha = -\frac{1}{\sqrt{g}} \partial_\beta \left(\sqrt{g} T^\beta_a S^a{}_\alpha\right) + p_a \partial_\alpha q^a + T^\beta_a \partial_\alpha R^a_\beta + \Pi_\phi \partial_\alpha \phi. \tag{9}$$

We note that due to the property $U_{ij}\partial_t U_{ik} = -U_{ik}\partial_t U_{ij}$ the time derivative of the matrix $U^a{}_b$ drops out from the expression (7) and besides, only $n\,(n-1)\,/2$ components of the matrix T^α_a (i.e., with $a > \alpha$) should be considered as independent functions which are canonically conjugate to the triangle matrix variables $R^a{}_\alpha$.

In the super-Hamiltonian H^0 constraint the quantities R^a_α and T^α_a contribute only to the spatial curvature in the term $U = g\,(W - R)$ and for the case of $n = 3$ the functions R^a_α are connected purely with transformations of the coordinate system and may be removed by solving the super-momentum constraints $H^\alpha = 0$ [3] (which expresses independent components of T^α_a as functions of p_a, q^a, Π_ϕ, and ϕ). In the multidimensional case, however, the functions R^a_α contain $\frac{n(n-3)}{2}$ dynamical functions as well which cannot be removed by coordinate transformations. However, in what follows we shall use model representations for the potential term U in which the dependence of U on R^a_α and T^α_a can be neglected (e.g., the generalized Kasner model (GKM) corresponds to the case where we neglect the spatial curvature, the inhomogeneous Mixmaster model (IMM) corresponds to the case where we replace the spatial curvature term with a set of infinite potential walls). Therefore in these models the super-Hamiltonian will not depend explicitly on R^a_α and T^α_a and these functions will have a passive character and can be considered separately. Indeed, in the case of the GKM or IMM the evolution of these functions is completely governed by the supermomentum constraint (9) which can be used to express n independent functions among R^a_α and T^α_a via the rest passive (R^a_α and T^α_a) and dynamical (p_a, q^a, Π_ϕ, ϕ) functions. We note that in the GKM all the passive functions represent merely constants of the motion (e.g., see Ref. [8]), while in the IMM the oscillatory evolution is accompanied by a rotation of the Kasner vectors which is completely determined by the momentum constraint (e.g., see Ref. [2, 22]).

After having shown how the above formal decomposition of the metric variables into scale functions and "reference" vectors acquires a precise dynamical meaning in the above action, we must make a key distinction among the scale functions themselves by extracting their isotropic component from the anisotropic ones. This is accomplished by further refining the metric parametrization by introducing coordinates on the space of scale variables which are quasi-orthonormal with respect to the DeWitt minisuperspace metric following Misner [4]

$$q^a = A^a{}_j \beta^j + \alpha \quad \beta^n = [n(n-1)/2]^{-1/2}\phi\,, \tag{10}$$

where $j = 1, ..., n-1$, and the suitably chosen constant matrix A^a_j obeys the conditions

$$\sum_a A^a{}_j = 0, \qquad \sum_a A^a{}_j A^a{}_k = n(n-1)\delta_{jk}\,, \tag{11}$$

for example

$$A^a{}_j = \sqrt{\frac{n(n-1)}{j(j+1)}}(\theta^a{}_j - j\delta^a{}_j), \qquad \theta^a{}_j = \begin{cases} 1, & j > a\,, \\ 0, & j \le a\,. \end{cases} \tag{12}$$

Since $g = \exp(n\alpha)$, we see that α corresponds to the volume or isotropy degree of freedom, while β^j ($j = 1, 2, ..., n-1$) describe the anisotropy of the model. When these variables are sufficiently suppressed in the sense that they asymptotically approach constant values, we may speak of quasi-isotropization of the model.

Then the action expressed in these variables formally resembles the action of a relativistic particle moving in a potential (here the index r runs from 1 to n)

$$I = \int \left(P_r \frac{\partial}{\partial t}\beta^r + P_\alpha \frac{\partial}{\partial t}\alpha - \frac{N}{n\,(n-1)\sqrt{g}} \left(\sum_r P_r^2 + \widetilde{U} - P_\alpha^2 \right) \right) d^n x dt, \tag{13}$$

where the potential term $\widetilde{U} = n\,(n-1)\,U$ may be viewed as a "mass term" for the particles which depends on dynamical variables (on the position in the phase space) and is not everywhere positive.

It is important to emphasize that since our goal is to consider simple models (GKM and IMM), for the sake of simplicity we removed from the above action the passive functions (the terms $T_a^\alpha \partial R_\alpha^a / \partial t$) as well as the super-momentum terms which as explained above can be considered separately.

3 Construction of the Inhomogeneous "Bridge" Model

In this section we derive a "generic" inhomogeneous solution connecting the Kasner-like behavior (to be regarded as one of the Kasner epochs in the oscillatory regime) with an inflationary regime [27]. It is worth noting that, although our analysis is done on a purely classical level and in the absence of ultrarelativistic matter, nevertheless from the qualitative point of view it has a predictive character even in a more general context. Indeed on the one hand we may expect that during the last Kasner epoch of the oscillatory regime the so-called *quantum potential*[4] plays, in the Hamilton-Jacobi equation, the role of a small correction to the classical potential (which can be modelled by a set of infinite potential walls, e.g. see the next section). From the other hand, either during the Kasner regime in the presence of a scalar field or during the inflationary phase, the contribution of the ultrarelativistic matter is negligible in comparison with the kinetic [28] and/or the potential [29] energy of the scalar field respectively.

The inflationary solution can be obtained from the action (13) if we impose restrictions of the form

$$\frac{1}{g}\widetilde{U} \simeq V(\phi) \simeq const \gg R\ ,\ g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \tag{14}$$

which can be realized by an appropriate process of spontaneous symmetry breaking as described in the standard literature on this subject (see [29], [30, 31] and [32]). The resulting model given below describes the inflationary expansion of an inhomogeneous universe in the limit $g \to \infty$ (i.e. $\alpha \to \infty$), while the Kasner-like regime [8] is obtained asymptotically approaching the singularity for $g \to 0$ (i.e. $\alpha \to -\infty$). To derive such a solution we use the Einstein-Hamilton-Jacobi method since in the sense mentioned above this theory is the quasiclassical approximation to quantum gravity and also because it is computationally convenient.

Let us consider the situation where in (13) the potential (mass) term can be approximated as

$$\widetilde{U} = n(n-1)\, g\Lambda, \tag{15}$$

where $\Lambda = \Lambda(x^i) \approx const$. These conditions impose peculiar restrictions on the degree of inhomogeneity of the scalar and gravitational fields and on the potential form of $V(\phi)$. Then the Einstein-Hamilton-Jacobi equations are

$$P_r = \frac{\delta I}{\delta\beta^r},\ P_\alpha = \frac{\delta I}{\delta\alpha},\ \ H^0(\alpha, \beta^r, P_\alpha, P_r, \Lambda) = 0 \tag{16}$$

[4] If we separate the wave functional of the universe into its modulus and its phase, then the latter will satisfy the Hamilton-Jacobi equation containing an additional "quantum potential".

or explicitly

$$\sum_r \left(\frac{\delta I}{\delta \beta^r}\right)^2 - \left(\frac{\delta I}{\delta \alpha}\right)^2 + n\,(n-1)\exp{(n\alpha)}\,\Lambda = 0. \tag{17}$$

The evolution of the residual variables remaining in the Kasner reference vectors $\ell^a{}_\alpha$ are determined by the super-momentum constraint equations $H_\alpha = 0$ and can be expressed via the functions (P_r, β^r).

The solution of eq. (16) can be expressed in the form

$$I(\beta^r, \alpha) = \int_S \left\{ K_r \beta^r + (\frac{2}{n} K_{\ \alpha} + \frac{K}{n} \ln \mid \frac{K_\alpha - K}{K_\alpha + K} \mid \right\} d^n x, \tag{18}$$

where $K_\alpha(K_r, \alpha) = \pm\sqrt{\sum_r K_r^2 + n\,(n-1)\,\Lambda \exp{(n\alpha)}}$, $K = \sqrt{\sum_r K_r^2}$ and K_r are arbitrary "constant" functions of the spatial coordinates (i.e. independent of the time variable). Here S denotes the whole available spatial domain in which restrictions (14) are fulfilled.

The signs $\pm$ before the square root correspond to the two possibilities for the variation of the local spatial volume in S (it depends on whether collapse or expansion of S is considered). Indeed, from (13) we find that the variation of α is determined by the Hamilton equation

$$\frac{\partial \alpha}{\partial t} = -\frac{2NP_\alpha}{n\,(n-1)\exp{(n\alpha/2)}} \tag{19}$$

and $N/\exp{(n\alpha/2)} > 0$. In order to further simplify our analysis, we choose as a time variable the quantity α (i. e, $\frac{\partial}{\partial t}\alpha = 1$), which implies the time gauge condition $N = n\,(n-1)\exp{(n\alpha/2)}\,/\,(-2P_\alpha))$ (since the lapse function should be positive by definition, we must have $P_\alpha < 0$).

Now according to the Hamilton-Jacobi method, we differentiate with respect to the quantities K^r and then by putting the results equal to arbitrary "constant" functions, we find the solutions describing the trajectories of the system ($\frac{\delta I}{\delta K^r} = \beta_0^r$)

$$\beta^r(\alpha, x^i) = \beta_0^r(x^i) + \frac{K_r}{n\,|K|} \ln \left| \frac{K_\alpha - K}{K_\alpha + K} \right| \ , \tag{20}$$

where $\beta_0^r(x^i)$ are arbitrary "constant" functions. In the asymptotic limit $g \to \infty$ (i.e. $\alpha \to \infty \ \Rightarrow \ K_\alpha \to \infty$) the solution (20) transforms into the inflationary solution obtained in [27], corresponding to the quasi-isotropization of the model since the parametric functions β^r approach (exponentially) the "constant" values $\beta_0^r(x^i)$. In the opposite limit $g \to 0$ (i.e. $\alpha \to -\infty \ \Rightarrow \ K_\alpha \to K$) (20) transforms into the generalized Kasner solution as it should, modified by the presence of the scalar field (see the [22])

$$\beta^r(\alpha, x^i) = \beta_0^r(x^i) - \frac{K_r}{K}(\alpha - \alpha_0) \ , \tag{21}$$

where α_0 renames the remaining constant terms.

We conclude our analysis by establishing the relation between our time variable α and the synchronous time T, which has a precise cosmological interpretation. These two variables are connected by the differential expression

$$dt = Nd\alpha = -\frac{n\,(n-1)\exp{(n\alpha/2)}}{2P_\alpha})d\alpha. \tag{22}$$

Now it is easy to see that from the Hamilton equation obtained by varying the action with respect to α we get, having fixed our time gauge (and remembering that $H^0 = 0$) the following asymptotic behaviors:

$$\alpha \rightarrow \infty \ : \ P_\alpha \propto -\sqrt{\Lambda}\exp{(n\alpha/2)} \ , \quad \alpha \rightarrow -\infty \ : \ P_\alpha \propto const < 0. \tag{23}$$

By substituting these relations into (22), we find the expected (familiar) asymptotic relations (which make evident the character of the two "opposite" regimes)

$$\alpha \rightarrow \infty \ : \ \alpha \propto \sqrt{\Lambda}t \Rightarrow \sqrt{g} \propto \exp\left(C_1\sqrt{\Lambda}t\right) \quad \alpha \rightarrow -\infty \ : \ \alpha \propto \frac{2}{n}\ln C_2 t \Rightarrow \sqrt{g} \propto t \tag{24}$$

where C_1 and C_2 denote two constant values.

The existence of this solution shows how the inflationary scenario can provide the necessary dynamical "bridge" between the fully anisotropic and the quasi-isotropic stage of the universe evolution.

4 Misner-Chitrè-like approach

Though our solution is perfectly characterized by the above Misner-like variables, nevertheless to make precise the restrictions to be imposed on the initial conditions for the existence of such an interpolating regime, it is necessary to investigate a bit in detail the finite oscillating evolution to the singularity and therefore it is much more convenient to make use of the so-called Misner-Chitrè-like variables. [4] In order to introduce these variables, the scale functions q^a may instead be parameterized as follows (see Ref. [13])

$$q^a = \ln R_0^2 + M_a \ln g; \ \sum_a M_a = 1 \ , \quad a = 1, 2, ..., n \tag{25}$$

where we distinguished a slowly varying function of time R_0, which characterizes the absolute value of amplitude of the metric functions [5, 25] and is specified by initial conditions (see below), from the anisotropy parameters M_a, which characterize the model's anisotropy; now the quantities $\ln g = \sum_a q^a - 2n \ln R_0$ and M_a can be expressed in terms of the new set of Misner-Chitrè-like variables τ and y^j $(j = 1, 2...(n-1))$, as follows:

$$\ln g = -ne^{-\tau}\frac{1+y^2}{1-y^2} \quad M_a\left(y^j\right) = \frac{1}{n}\left(1 + \frac{2y^j A^a{}_i}{1+y^2}\right) \ , \tag{26}$$

where $A^a{}_i$ is the matrix (12). The Misner variables β^j are related to y^i by

$$\beta^j = -e^{-\tau}2y^j/\left(1-y^2\right), \ \ \alpha = \ln R_0^2 - e^{-\tau}\left(1+y^2\right)/\left(1-y^2\right) \ .$$

The parameterization (26) is defined within the domain $-\infty < \tau < \infty$, $0 < y < 1$ ($y \equiv \sqrt{\sum_j (y^j)^2}$) and (with $0 \leq g \leq 1$) an appropriate choice of the function R_0 allows one to cover all of the classically allowed region of the configuration space using this parameterization.

Within this choice of variables, the evolution of the scale functions is described by the action

$$I = \int \left\{ \left(\vec{P}_{\vec{y}} \frac{\partial \vec{y}}{\partial t} + h \frac{\partial \tau}{\partial t} + P_n \frac{\partial \beta^n}{\partial t} \right) - \frac{N e^{2\tau}}{n\,(n-1)\, R_0^n \sqrt{g}} \left[\varepsilon^2 - h^2 + \Pi + e^{-2\tau} P_n^2 \right] \right\} d^n x dt, \tag{27}$$

where $\vec{P}_{\vec{y}}$ and h denote respectively the conjugate momenta to $\vec{y}$ and τ, while $\varepsilon^2 = \frac{1}{4} \left(1 - y^2\right)^2 \vec{P}_{\vec{y}}^2$ and the potential term Π has the following structure

$$\Pi = n\,(n-1)\, R_0^{2n-2} e^{-2\tau} \sum_{a,b,c} \lambda_{abc} g^{1+M_a-M_b-M_c} + n\,(n-1)\, R_0^{2n} g e^{-2\tau} V(\beta^n). \tag{28}$$

Here the coefficients λ_{abc} (constructed by the spatial derivatives of the reference vectors ℓ_α^a) are slow functions of $\ln g$, i.e. of the time variable, and characterize the initial intensity of the inhomogeneity field. When $g \ll 1$, we can use the approximation of deep oscillations [1, 2] [5], in which the above potential is modeled by a set of potential walls, each of them having the form

$$g^{\sigma_a} \rightarrow \theta_\infty[\sigma_a] = \begin{cases} +\infty\,, & \sigma_a < 0, \\ 0\,, & \sigma_a > 0, \end{cases} \tag{29}$$

As a result the whole potential becomes asymptotically ($\Pi \rightarrow \Pi_\infty$) independent of Kasner vectors, i.e. $\Pi_\infty = \sum \theta_\infty\left(\sigma_a\right)$.

By solving the Hamiltonian constraint $H^0 = 0$ in (27) we define the ADM action [24], reduced to the physical phase space, by a standard procedure which leads to the following (reduced) action

$$I_{red} = \int (\vec{P}_{\vec{y}} \cdot \frac{d\vec{y}}{d\tau} + Q \frac{\partial}{\partial \tau} q - H_{ADM}) d^n x d\tau, \tag{30}$$

where

$$H_{ADM} \equiv -h = \sqrt{\varepsilon^2 + e^{-2\tau} Q^2 + \Pi} \tag{31}$$

is the ADM Hamiltonian and τ now plays the role of the time variable ($\dot{\tau} = 1$), once the gauge is fixed by $N_{ADM} = \frac{n(n-1) R_0^n \sqrt{g}}{2 H_{ADM}} e^{-2\tau}$. For convenience we have also redefined the scalar field variables as follows $q = \beta^n$ and $Q = P_n$.

[5] It is a well-known result that, close enough to the singularity, the inhomogeneous Mixmaster evolution can be represented as a sequence of Kasner epochs (i.e. completely neglecting the potential term), while the transition from one epoch to the next one can be regarded as an instantaneous phenomenon (i.e. instant bounces against the potential walls). This well-established oscillating behavior is called the "deep oscillation approximation" and corresponds in our Hamiltonian picture to the replacement of the actual potential term with a set of infinite potential walls.

Thus, in the limit $\tau \to -\infty$, which corresponds to $\Pi \to \Pi_\infty(y^j)$ (the independence of the potential walls on the time variable τ is the most relevant advantage in using Misner-Chitrè-like variables), the system (30) is nothing more than a point-like realization of a billiard on the $(n-1)$ −dimensional Lobachevsky space. By other words, in each point of the space, the system dynamics is isomorphic to the motion of a point particle on the (negative constant curvature) $(n-1)$-dimensional hypersurface formed by all the admissible values y^j (the real gravitational degrees of freedom). On this domain the potential walls cut a region, which in dimensions $n \leq 9$ has a finite volume[6] and therefore (in the absence of a scalar field) the resulting billiard exhibits strong mixing properties [4]. The role of scalar field (whose momentum, in this approximation, is simply an arbitrary "constant" function $Q(x^i)$) in the evolution of metric functions consists of making geodesic lines on the billiard to be of a finite length and therefore of suppressing the chaotic regime (no other bounces against the potential walls take place).

Thus, with this scheme in our hands, we are now able to easily specify the appropriate inequalities characterizing different dynamical regimes. When approaching the singularity, the chaoticity can develop only in those regions of the universe where the energy of the scalar field is sufficiently small $e^{-2\tau}Q^2 \ll \varepsilon^2$. We also observe that the condition for applying the approximation (29) corresponds to free motion in the allowed domain and therefore it can be written as follows

$$\varepsilon^2 \gg U. \tag{32}$$

From the condition that the approximation of deep oscillations (29) breaks at the moment $g \sim 1$ ($\tau \sim 1$), we find that the function R_0 should be chosen as follows $R_0^{2n-2} = \frac{\varepsilon^2}{n(n-1)\lambda^2} e^{2\tau}$ (where $\lambda^2 = |\sum \lambda_{abc}|$), so that in fact the inequality (32) then just becomes $g \ll 1$ ($\tau \ll 1$).

5 Dependence of the Universe Evolution on the Initial Conditions

In this section we develop a synthesis of the interpolating regime constructed above, but in view of the different dynamical issues can take the universe in consequence of different initial conditions. All the considerations presented in the discussion below can be easily derived from the Hamiltonian function and equations associated with the reduced action I_{red}.

As emphasized above [22], in the presence of a scalar field, the final stage of the cosmological collapse ends with a monotonic Kasner-like behavior, i.e. the number of oscillations is always finite. From a phenomenological standpoint, if we consider an isotropic universe, this means that the effective "equation" of state for the gravitational waves (or the anisotropy parameters) is slightly softer then the one for the scalar field

[6]In the presence of dilaton-p-form fields billiards have finite volumes in all dimensions, e.g., see Refs. [19, 20].

(behaving like stiff matter, characterized by $\varepsilon = p$), i.e., $\varepsilon \gtrsim p$. To see this explicitly, we may use some of the results derived above.

Indeed near the singularity we have $\Pi \to \Pi_\infty(y)$ and the scalar field momentum does not depend explicitly on the time variable $Q = const$ and therefore the same behavior characterizes the ADM "energy" of the anisotropy $\epsilon_h^2 = \epsilon^2(\vec{y}, \vec{P}_{\vec{y}}) = const$ (clearly from its expression, this quantity does not depend explicitly on τ). However, the ADM energy of the scalar field depends on time $\epsilon_q^2 = (Q)^2 e^{-2\tau}$ and we get the relation $\epsilon_q/\epsilon_h \sim e^{-\tau} \sim \ln g$. Thus in the limit $\tau \to -\infty$ ($g \to 0$) the scalar field dominates and the ADM Hamiltonian does not depend on the gravitational variables $H_{ADM} \to \epsilon_q$, i.e. it turns out $\vec{y} = const$, $\vec{P}_{\vec{y}} = const$ (this corresponds to a stable Kasner regime approaching the singularity).

On the other hand, during the expansion of the universe the role played by the scalar field through the evolution becomes smaller and smaller an, therefore the following various types of dynamical regimes can take place.

5.1 The vacuum-type regime

If the initial ADM energy of the scalar field is not very big, then the ADM energy of the anisotropy starts to dominate and this type of regime will be described by the oscillatory behavior with a frozen scalar field $Q = const$, $q = const$. The scalar field potential, which is in general always negligible near the singularity (i.e. at sufficiently high temperatures), in this case behaves like an effective cosmological constant $V(q) = const$. However if this constant remains small as compared to the ADM energy of the anisotropy during the whole period of applicability of the BKL picture, then this would result in the vacuum-type evolution.

To define the moment $\tau_i n$ when the BKL approximation breaks down (i.e., $g \sim 1$), we consider the synchronous cosmological time, which is related to τ by the equation

$$dt = N_{ADM} d\tau = \frac{n(n-1) R_0^n \sqrt{g}}{2H_{ADM}} e^{-2\tau} d\tau. \tag{33}$$

In the vacuum stage we have $H_{ADM} = \epsilon_h = const$, which gives us

$$\sqrt{g} \sim \frac{t/t_{in}}{1 - \ln(t/t_{in})}, \tag{34}$$

where $t_{in} = \frac{2(n-1)R_0^n}{n\epsilon_h}$ and $R_0^{2n-2} = \frac{\varepsilon^2}{n(n-1)\lambda^2} e^{2\tau}$. We recall that during the expansion, R_0 remains a constant in the oscillatory BKL regim, since as shown in Ref. [3, 4] $\lambda \sim \lambda_0 e^{\tau}$.

From the physical standpoint the moment τ_{in} ($g_{in} \sim 1$) corresponds to the case where the characteristic scale of the inhomogeneity L_{in} matches the horizon size L_h, $L_i \sim L_h$, the so-called moment of origin of a stable background e.g., see [17].

5.2 The inflationary-type regime

This kind of regime is realized when the scalar field potential grows until it is comparable with the ADM energy of anisotropy before the end of the BKL oscillatory regime. This dynamical feature results in the so-called inflationary-like evolution, replacing the last Kasner epoch of the oscillatory regime (recall that if the scalar field potential remains small, then the evolution corresponds to the vacuum case).

Under the assumption that the potential term $gV(q)$ starts to dominate before the end of the Mixmaster $\tau < \tau_{in} \sim 1$, then the ADM energy can be approximated by a function of the form

$$H_{ADM} \approx \sqrt{\epsilon_h^2 + n(n-1) R_0^{2n} g e^{-2\tau} V(q)} \tag{35}$$

(where we neglect the kinetic term of the scalar field and the spatial curvature term also). Thus the conditions for this regime to exit are expressed via the inequalities

$$\epsilon_q^2 \ll \epsilon_h^2 \sim n(n-1) R_0^{2n} g_c e^{-2\tau_c} V(q) \ , \tag{36}$$

where τ_c corresponds to the beginning of the inflationary evolution, i.e. the moment at which both terms at right hand side of (36) have the same order, that is to say

$$\epsilon_h^2 \sim \ n(n-1) R_0^{2n} g_c e^{-2\tau_c} V(q)$$

and the inequality which expresses the applicability of the Mixmaster approximation reads $g_c \ll g_{in} \sim 1$, or equivalently, in terms of the synchronous cosmological time, as follows from (34)

$$t_c \ll t_{in} \ , \tag{37}$$

since t_c is given by

$$t_c = \sqrt{\frac{n-1}{nV(q)}}. \tag{38}$$

From a physical point of view the last inequality expresses the well known condition $L_i \gg L_C$ for the inflationary dynamics, where L_C is related to the field "Compton scale". We also recall that at later times, i.e., as $t > t_c$, the inflationary evolution leads to the scalar curvature term in H_{ADM} overtaking the kinetic energy of the anisotropy ϵ_h^2. However, we emphasize that both these terms will remain very small compared to the effective cosmological constant $V(q)$.

We note that the inflationary regime has a finite duration, due to a 'slow' evolution of the scalar field (the so-called *slow-rolling phase*), which is well described in the canonical literature on inflation [7]. The slow evolution phase of the scalar field can be found from the Hamiltonian equations

$$\frac{\partial}{\partial \tau} q = \frac{Q e^{-2\tau}}{H_{ADM}} \approx 0, \tag{39}$$

and

$$\frac{\partial}{\partial \tau} Q = -\frac{1}{2} \frac{n(n-1) g e^{-2\tau} R_0^{2n} V(q)'}{H_{ADM}}. \tag{40}$$

Hence in the synchronous gauge $(N = 1)$ $dt/d\tau = \frac{n(n-1)}{2} {R_0}^n \sqrt{g} \exp(-2\tau)/H_{ADM}$ we get

$$\frac{\partial}{\partial t} q = \frac{2Q}{n(n-1) R_0^n \sqrt{g}}, \tag{41}$$

$$\frac{\partial}{\partial t}Q = -R_0^n \sqrt{g} V(q)', \qquad (42)$$

where the initial conditions should be chosen so that

$$Q_0^2 \ll \epsilon_h^2 e^{2\tau_c} \lesssim n(n-1) R_0^{2n} g_c V(q_c) \ . \qquad (43)$$

The rate of expansion is described by the equation $dt/d\tau$ which can be rewritten as follows (we take $\ln g = -ne^{-\tau}$ $(y = 0)$ and $H_{ADM} = \sqrt{n(n-1) R_0^{2n} g e^{-2\tau} V(q)}$)

$$\frac{\partial}{\partial t}\sqrt{g} = \sqrt{\frac{nV(q)}{(n-1)}}\sqrt{g}, \ \sqrt{g} \sim \sqrt{g_c} \exp\left(\int_{t_c}^{t} \sqrt{\frac{nV(q)}{(n-1)}} dt\right). \qquad (44)$$

5.3 The scalar-field-dominated-type regime

Finally we consider another type of regime which takes place when the scalar field dominates during the entire Mixmaster approximation, i.e. $\epsilon_q^2 \gg \epsilon_h^2$. In this case the solution has a monotonic behavior; we can neglect the anisotropy functions from the very beginning and the evolution proceeds in the same way as in the case of isotropic models in the presence of a scalar field.

5.4 Brief concluding remarks

We conclude this section by emphasizing some relevant features of the interpolating solution we have obtained which help in getting physical insight into its cosmological setting.

i) It is remarkable that the condition ensuring the inflationary scenario starts well inside the range of validity of the oscillatory regime is nothing more than the assumption at that moment τ_c that the "Compton length" associated with the scalar field L_c be much smaller than the inhomogeneous scale L_{in} of the universe; this condition is telling us that, as in the homogeneous case (when the condition for the existence of an inflationary stage reads in this same form), even in this general context the inflation can start only if the spatial gradients are sufficiently small. Thus this requirement for the validity of the BKL approach which is at the heart of our interpolating approach is just the inhomogeneous generalization of the well-know standard restriction for homogeneous inflation; this fact provides very strong physical support for the dynamical analysis developed here.

However, we must mention that the condition so obtained is a point-like one and in principle, cannot take place in some spatial domain. It is worth noting that since we require that inflation starts before the BKL regime ends, i.e. when the horizon size is still much less then the inhomogeneous scale ($L_h \ll L_i$), the dynamical regime derived above can (nevertheless) occur naturally in causal regions.

ii) An important feature of the inflationary scenario consists of the frozen dynamics acquired by the anisotropy of the universe. In other words, during this stage of the evolution, the functions β_+ and β_- remain almost constant. As a consequence, the only evolving variable is now *alpha*, i.e. the volume of the universe; this fact makes it evident that the dominant term during the inflation is $\Lambda e^{3\alpha}$ and any other term in the spatial curvature, although increasing like (at most) $e^{2\alpha}$, becomes more and more negligible.

iii) Finally we emphasize that the results obtained above do not depend on the particular form taken by the scalar field potential, as long as it realizes a spontaneous symmetry breaking process and a sufficiently long phase of slow-rolling [33].

It is also worth noting that the compatibility of the inflationary scenario with our actual cosmic phenomenology seems to be confirmed by recent observations on the microwaves background radiation [34], thus allowing, on the basis of the analysis developed here, very general dynamical behavior for the primordial phases of the universe evolution.

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